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On the Mean Pair Correlation Function of $\pm J$ Ising Spin Glasses

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We provide a formula and an upper bound for the average over the disorder of the pair correlation function of $\pm J$ Ising spin glasses by using the symmetries of the system. We show the decay of the mean spin pair correlation function when the proportion of antiferromagnetic bonds is larger than the critical parameter associated with the pair dissociation phase transition.

KEY WORDS: Spin glasses; local gauge invariance; percolation; random interaction random cluster measure.

1. INTRODUCTION

We consider the $\pm J$ Ising spin glass on a finite graph $G=(V, E)$ with node set V and edge set E, and random Hamiltonian given by

$$H_y(\sigma)=\sum_{(ij)\in E}(-1)^{y_{ij}}\sigma_i\sigma_j, \qquad \sigma\in\Omega:=\{-1,+1\}^V \tag{1.1}$$

where the random vector $y\in\{0,1\}^E$ represents the disorder, and is distributed according to a product Bernoulli measure $\mu_p^{\otimes}(y)=p^{|y|}(1-p)^{|E|-|y|}$, where $|y|$ is the Hamming weight of y, for some $0\leqslant p\leqslant 1/2$. Given $p_b\in[0,1]$, let β_{p_b} be given by the relation

$$p_b=1-\exp(-\beta_{p_b})$$

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and consider the Gibbs probability measure on Ω

$$\pi^{y}_{\beta_b}(\sigma) := Z_{\beta_b}(y)^{-1} \exp\left(\frac{\beta_{p_b}}{2} H_y(\sigma)\right) \tag{1.2}$$

where $Z_{\beta_b}(y)$ denotes the partition function of the system. Moreover let $\hat{p}_b := 1 - \sqrt{1 - p_b}$, with $\beta_{\hat{p}_b} = \beta_{p_b}/2$. The aim of this note is to give some properties of the pair correlation function $\langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}}$, where $\langle \cdot \rangle_{\pi^y_{\hat{p}_b}}$ denotes the average under Gibbs measure, by considering its average over the disorder

$$\mathrm{Ave}_{\otimes P}(\langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}}) = \sum_{y \in \{0,1\}^E} \mu^{\otimes}_{p}(y) \langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}}$$

Our approach considers the Fortuin–Kasteleyn representation of the covariance by the random cluster measure (see Okiji and Kasai (1988) or Newman (1990)). Given $y \in \{0, 1\}^E$, consider the event in $\Omega \times \{0, 1\}^E$ given by

$$\tilde{U}_y := \{(\sigma, (n));\ n_{ij}(-1)^{y_{ij}} \sigma_i \sigma_j \geqslant 0,\ \forall (ij) \in E\}$$

and

$$U_y := \{(n) \in \{0, 1\}^E;\ \exists \sigma \in \Omega \text{ with } (\sigma, (n)) \in \tilde{U}_y\}$$

the family of unfrustrated bond configurations. The FK probability measure is then given by

$$\mu^{y}_{p_b}((n)) := Z_{n, p_b}(y)^{-1}\, 2^{\|(n)\|} \mu^{\otimes}_{p_b}((n))\, I_{U_y}((n)) \tag{1.3}$$

where $Z_{n, p_b}(y) = \exp(-|E|\, \beta_{\hat{p}_b})\, Z_{\hat{p}_b}(y)$ is the partition function, $\|(n)\|$ is the number of clusters contained in the bond configuration (n), and I_{U_y} is the indicator function of the event U_y. Let

$$\eta_{ij}((n)) := I(i \rightsquigarrow j) \prod_{e \in \gamma} (-1)^{y_e}$$

where $i \rightsquigarrow j$ means that i and j are linked by a path γ of occupied bonds in (n) $(n_e = 1,\ e \in \gamma)$, the product $\prod_{e \in \gamma}(-1)^{y_e}$ being independent of the path when (n) is unfrustrated. The following relations are well known (see, e.g., N. (1990), Coniglio, Liberto, Monroy and Peruggi (1991) or Gandolfi, Keane and Newman (1992))

$$\langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}} = \mathrm{E}_{\mu^y_{p_b}}(\eta_{ij}) \tag{1.4}$$

with

$$\mu_{p_b}^{y}(i \rightsquigarrow j) = \mu_{p_b}^{y}(\eta_{ij} = 1) + \mu_{p_b}^{y}(\eta_{ij} = -1)$$

and

$$\langle \sigma_i \sigma_j \rangle_{\pi_{\beta_b}^{y}} = \mu_{p_b}^{y}(\eta_{ij} = 1) - \mu_{p_b}^{y}(\eta_{ij} = -1)$$

In a previous work (Mazza (1997)), we have considered the associated percolation problem by using the local gauge invariance of the system by looking at a special line of the phase diagram, the so-called Nishimori line, which is given by the relation $\hat{p}_b = p^*$, where $p^* := 1 - \sqrt{p/(1-p)}$, for $0 \leqslant p < 1/2$, $\tanh(\beta_{p^*}) = 1 - 2p$ (Nishimori (1980)). We proved that $\text{Ave}_{\otimes p}(\mu_{p_b}^{y}) \succ \mu_{h(p_b)}^{\otimes}$, $h(p_b) := p_b/(2 - p_b)$, where $\succ$ denotes stochastic domination, yielding the critical value $p = (1 - p_c(\mathbf{Z}^d))/2$, $G = \mathbf{Z}^d$, where $p_c(\mathbf{Z}^d)$ is the critical bond probability associated with independent percolation on $\mathbf{Z}^d$, $d \geqslant 2$. Gandolfi (1997) observed in fact that the proof given in (M. (1997)) gives the identity $\text{Ave}_{\otimes p}(\mu_{p_b}^{y}) = \mu_{h(p_b)}^{\otimes}$, $\hat{p}_b = p^*$, and extended this result to systems with fixed boundary conditions. Comparison with the phase diagrams of the binary tree (Carlson *et al.* (1990)) and of the cactus tree (Katsura (1977)) reveals the existence of a domain of the (p, T_b) diagram, $T_b := \beta_{p_b}^{-1}$, where percolation occurs but $\text{Ave}_{\otimes}(\langle \sigma_i \sigma_j \rangle_{\pi_{\beta_b}^{y}}) \to 0$ as $\|j - i\| \to \infty$. The nature of this phase is unclear, and is under current numerical investigation (see, e.g., Cataudella *et al.* (1991, 1994) or Imaoka, Ikeda and Kasai (1997)). In this work, we use the symmetries of the problem to give a formula for the mean pair correlation function as well as an upper bound of the form

$$|\text{Ave}_{\otimes p}(\langle \sigma_i \sigma_j \rangle_{\pi_{\beta_b}^{y}})| \leqslant \langle \sigma_i \sigma_j \rangle_{\pi_{p^*}^{0}} \text{Ave}_{\otimes p}(\mu_{p_b}^{y}(i \rightsquigarrow j)), \qquad p < 1/2 \tag{1.5}$$

The above inequality shows the appearance of a Gibbs measure having the same Hamiltonian but viewed at a new temperature: the average of the pair correlation function is bounded by the product of a ferromagnetic pair correlation function under a temperature of disorder $\beta_{p^*}^{-1}$ given by $\tanh(\beta_{p^*}) = 1 - 2p$, with the average over the disorder of the FK mass $\mu_{p_b}^{y}(i \rightsquigarrow j)$. This implies the decay of the mean spin pair correlation function when the amount of disorder is larger than the critical parameter associated with the pair dissociation phase transition.

2. PRELIMINARIES

Given $V_1 \subset V$, the set of edges $c \subset E$ each of which is incident at a node in V_1 and at a node of $V_2 = V \backslash V_1$ is called a cut in G. The set of such cuts is denoted by $\mathscr{C}$. Let $GF(2)$ be the field of two elements. To every subset $S \subset E$ of edges is associated a binary vector $f_S \in GF(2)^{|E|}$, called the characteristic vector of S, given by $f_S(e) = 0$ if $e \notin S$ and $f_S(e) = 1$ otherwise. The set of characteristic vectors associated with the cuts form a linear subspace $\mathscr{C}$ of $GF(2)^{|E|}$ of dimension $|V| - |G|$, where $|G|$ denotes the number of connected components of G. In the remaining, we assume that $|G| = 1$. The incidence matrix of the graph is the $|V| \times |E|$ binary matrix D_G in which rows $i = 1,..., |V|$ are the characteristic vectors associated to the elementary cuts $c(i) \in \mathscr{C}$, that is the cuts associated with characteristic vectors f_{S_i}, where S_i denotes the set of edges incident to i. D_G has rank $|V| - 1$, the relation being given by $\sum_{i \in V} c(i) = 0 \bmod 2$. Every subset of $|V| - 1$ rows of D_G forms a basis of $\mathscr{C}$. In this work, we see $\mathscr{C}$ as a linear code over $GF(2)$, the cut code of G. (see, e.g., Biggs (1992)).

We next recall some definitions and results given in M. (1997). A function $f: GF(2)^{|E|} \to \mathbf{R}$ is said to be gauge invariant when $f(y + c) = f(y)$, $\forall y \in GF(2)^{|E|}$, $c \in \mathscr{C}$. For example $\max_{\sigma \in \Omega} H_y(\sigma)$, $Z_{\beta_b}(y)$ and the FK measure $\mu^y_{p_b}((n))$ on a fixed subgraph (n) are gauge invariant. For such functions,

$$\mathrm{Ave}_{\otimes p}(f) = \sum_{t \in \mathscr{T}} f(t) \sum_{c \in \mathscr{C}} \mu^{\otimes}_p(t + c)$$

for every linear supplement $\mathscr{T}$ of $\mathscr{C}$. The fact of choosing a particular linear supplement $\mathscr{T}$ is called gauge fixing. Let $\bar{\mu}^{\otimes}_p$ be the marginal of $\mu^{\otimes}_p$ on $\mathscr{T}$. Then, if f is gauge invariant

$$\mathrm{Ave}_{\otimes p}(f) = \sum_{t \in \mathscr{T}} f(t)\, \bar{\mu}^{\otimes}_p(t)$$

Let $z_p := p/(1 - p)$, $p < 1$. Then $\bar{\mu}^{\otimes}_p(t) = (1 - p)^{|E|} \sum_{c \in \mathscr{C}} z_p^{|t + c|}$. Given $q \leqslant 1$ and β_q given by $q = 1 - \exp(-\beta_q)$, $\beta_q \in \mathbf{R}$, let $Z_q(y)$ be the Ising partition function associated with the disorder y and real inverse temperature β_q. Given $p \in [0, 1[$, let

$$p^* := 1 - \sqrt{\frac{p}{1 - p}}, \qquad p^* = 1 - \exp(-\beta_{p^*}), \qquad \tanh(\beta_{p^*}) = 1 - 2p \tag{2.1}$$

Then we have Fradkin–Hubermann–Shenker (1978) relation (see, e.g., M. (1997))

$$Z_{p^*}(t) = \cosh(\beta_{p^*})^{|E|}\, 2^{|E| + 1} \bar{\mu}^{\otimes}_p(t) \tag{2.2}$$

The Nishimori line is given by the relation $\hat{p}_b = p^*$, $p < 1/2$, where we recall that $\hat{p}_b = 1 - \sqrt{1 - p_b}$. On this line $\mathrm{Ave}_{\otimes p}(\mu^y_{p_b}) = \mu^{\otimes}_{h(p_b)}$, where

$h(p_b) = p_b/(2 - p_b)$ (see M. (1997) or G. (1997)). When G is bipartite, the same identity holds when $\hat{p}_b = (1-p)^*$, $p > 1/2$.

Let g be a spanning tree of G. Let T_g be the linear space over $GF(2)^{|E|}$ generated by the characteristic vectors associated with the edges of $E \backslash E(g)$, taken one by one, where $E(g)$ denotes the edge set of g. Then T_g is a linear supplement of $\mathscr{C}$, with $t \in \mathscr{T}_g \Rightarrow t_e = 0$, $\forall e \in g$.

Given a linear code $\mathscr{L}$ of $GF(2)^{|E|}$ and $s \in \mathbf{R}$, consider the weight enumerator $W_{\mathscr{L}}(s) := \sum_{w \in \mathscr{L}} s^{|w|}$. Consider the inner product $\langle x, x' \rangle := \sum_{e \in E} x_e x'_e \bmod 2$, $x, x' \in GF(2)^{|E|}$; the dual of any linear code $\mathscr{L}$ is given by

$$\mathscr{L}^{\perp} := \{ w \in GF(2)^{|E|}; \langle w, l \rangle = 0 \ \forall l \in \mathscr{L} \}$$

For example, the dual of the cut code consists in all Eulerian subgraphs of G, that is the set of all subgraphs in which node have even degrees. One important fact is MacWilliams identity (see, e.g., Roman (1992))

$$W_{\mathscr{L}^{\perp}}(s) = \frac{(1+s)^{|E|}}{|\mathscr{L}|} W_{\mathscr{L}}\left(\frac{1-s}{1+s}\right) \tag{2.3}$$

A basic example is van der Waerden's Eulerian expansion of the ferromagnetic partition function (van der Waerden (1941)). Using (2.2), we get that

$$\begin{aligned} Z_{p^*}(0) &= \cosh(\beta_{p^*})^{|E|}\, 2^{|E|+1} \bar{\mu}_p^{\otimes}(0) \\ &= \cosh(\beta_{p^*})^{|E|}\, 2^{|E|+1} (1-p)^{|E|} \sum_{c \in \mathscr{C}} z_p^{|c|} \end{aligned}$$

and thus, using (2.3), we arrive at

$$W_{\mathscr{C}}(z_p) = \frac{|\mathscr{C}|}{(2(1-p))^{|E|}} W_{\mathscr{C}^{\perp}}(1-2p), \qquad z_p = (1-s)/(1+s), \qquad s = 1-2p$$

and thus

$$Z_{p^*}(0) = 2^{|V|} \cosh(\beta_{p^*})^{|E|}\, W_{\mathscr{C}^{\perp}}(\tanh(\beta_{p^*}))$$

which is precisely van der Waerden's expansion, which is similar to the classical contour expansion of Peierls argument. Concerning $Z_{p^*}(t)$, $t \in \mathscr{T}$, consider the linear code $\mathscr{C}_t$ obtained by taking the direct sum $\mathscr{C} \oplus \langle t \rangle$, where $\langle t \rangle$ is the linear code generated by the singleton $\{t\}$. We can write

$$Z_{p^*}(t) = \cosh(\beta_{p^*})^{|E|}\, 2^{|E|+1}(1-p)^{|E|} \sum_{c \in \mathscr{C}} z_p^{|t+c|}$$
$$= \cosh(\beta_{p^*})^{|E|}\, 2^{|E|+1}(1-p)^{|E|}\, (W_{\mathscr{C}_t}(z_p) - W_{\mathscr{C}}(z_p))$$
$$= 2\cosh(\beta_{p^*})^{|E|}\, |\mathscr{C}|\, (2W_{\mathscr{C}_t^\perp}(1-2p) - W_{\mathscr{C}^\perp}(1-2p))$$

But $\mathscr{C}_t^\perp = \{w \in \mathscr{C}^\perp;\ \langle w, t\rangle = 0\}$, and it follows that

$$Z_{p^*}(t) = 2^{|V|}\cosh(\beta_{p^*})^{|E|} \sum_{w \in \mathscr{C}^\perp} \tanh(\beta_{p^*})^{|w|}\,(-1)^{\langle w, t\rangle}$$

For completeness, we give a short proof of the identity $\mathrm{Ave}_{\otimes p}(\mu^y_{p_b}) = \mu^{\otimes}_{h(p_b)}$, $p^* = \hat{p}_b$, $p < 1/2$. By definition, using the gauge invariance of the FK measure,

$$\mathrm{Ave}_{\otimes p}(\mu^y_{p_b}((n))) = 2^{\|(n)\|}\mu^{\otimes}_{p_b}((n)) \sum_{t \in \mathscr{T}} \bar{\mu}^{\otimes}_p(t)\, \frac{I_{U_t}((n))}{Z_{n,\,p_b}(t)}$$
$$= \frac{\exp(\beta_{\hat{p}_b}|E|)}{\cosh(\beta_{\hat{p}_b})^{|E|}\, 2^{|E|+1}}\, 2^{\|(n)\|}\mu^{\otimes}_{p_b}((n)) \sum_{t \in \mathscr{T}} I_{U_t}((n))$$

where we have used (2.2), $Z_{n,\,p_b}(t) = \exp(-|E|\beta_{\hat{p}_b})\, Z_{\hat{p}_b}(t)$, and $p^* = \hat{p}_b$. Using the gauge invariance of $I_{U_y}((n))$, we can write $\sum_{t \in \mathscr{T}} I_{U_t}((n)) = 1/|\mathscr{C}| \sum_{y \in GF(2)^{|E|}} I_{U_y}((n))$. But $(n) \in U_y$ if and only if every cycle of (n) contains an even number of edges of y, taken as subgraph, that is iff y is orthogonal to every cycle w of $\mathscr{C}(n)^\perp$, where $\mathscr{C}(n)$ denotes the cut code of the subgraph (n). Thus $(n) \in U_y$ if and only if the restriction of y to (n) is element of $\mathscr{C}(n)$. It follows that

$$\sum_{t \in \mathscr{T}} I_{U_t}((n)) = \frac{1}{|\mathscr{C}|}\, 2^{|E|-n_1}\, |\mathscr{C}(n)|$$

where n_1 is the number of edges of (n). We thus get that

$$\mathrm{Ave}_{\otimes p}(\mu^y_{p_b}((n))) = \frac{2^{\|(n)\|}\mu^{\otimes}_{p_b}((n))\, 2^{n_0}\, |\mathscr{C}(n)|}{2^{|V|}(2-p_b)^{|E|}}$$

$|\mathscr{C}(n)| = |V(n)| - |(n)|$, where $|(n)|$ is the number of connected components of (n) and $V(n)$ is the node set of (n). Using the fact that the number $i(n)$ of isolated points in (n) is such that $i(n) = |V| - |V(n)| = \|(n)\| - |(n)|$, we get that $\mathrm{Ave}_{\otimes p}(\mu^y_{p_b}((n))) = \mu^{\otimes}_{h(p_b)}$, $h(p_b) = p_b/(2-p_b)$, as required.

Let $p > 1/2$, and assume that G is bipartite. Then $\underline{1} \in \mathscr{C}$, $\underline{1} = (1,..., 1)^T$, and, from gauge invariance, $Z_{p^*}(y) = Z_{p^*}(y + \underline{1}) = Z_{(1-p)^*}(y)$, where $(1-p)^* = 1 - \sqrt{(1-p)/p}$, $\beta_{(1-p)^*} = -\beta_{p^*}$, and the same identity holds on the line $(1-p)^* = \hat{p}_b$.

3. MEAN PAIR CORRELATION FUNCTION

Given $i \neq j \in V$, let C_{ij} be the family of subgraphs $(n) \in \{0, 1\}^E$ containing some path of occupied edges ρ_{ij} linking i and j, which is assumed to be simple (no loops). Given such a path, let g_{ij} be any spanning tree of G containing ρ_{ij}, and let $\mathscr{T}_{ij}$ be the associated linear space with $\mathscr{T}_{ij} \oplus \mathscr{C} = GF(2)^{|E|}$ (see Section 2).

Proposition 1. Let $p^* = 1 - \sqrt{p/(1-p)}$, $p < 1$. Then

$$\mathrm{Ave}_{\otimes p}(\langle \sigma_i \sigma_j \rangle_{\pi^y_{p_b}}) = \sum_{(n) \in C_{ij}} 2^{\|(n)\|} \mu^{\otimes}_{p_b}((n)) \sum_{t \in \mathscr{T}_{ij}} \frac{I_{U_t}((n))}{Z_{n,\, p_b}(t)} \bar{\mu}^{\otimes}_p(t) \langle \sigma_i \sigma_j \rangle_{\pi^t_{p^*}}$$

$$|\mathrm{Ave}_{\otimes p}(\langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}})| \leqslant \langle \sigma_i \sigma_j \rangle_{\pi^0_{p^*}} \mathrm{Ave}_{\otimes p}(\mu^y_{p_b}(i \rightsquigarrow j)), \qquad 0 \leqslant p \leqslant 1/2$$

and

$$|\mathrm{Ave}_{\otimes p}(\langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}})| \leqslant \langle \sigma_i \sigma_j \rangle_{\pi^0_{(1-p)^*}} \mathrm{Ave}_{\otimes p}(\mu^y_{p_b}(i \rightsquigarrow j)), \qquad 1/2 \leqslant p < 1$$

Proof.

$$\mathrm{Ave}_{\otimes p}(\langle \sigma_i \sigma_j \rangle_{\pi^y_{\hat{p}_b}}) = \sum_{(n) \in C_{ij}} 2^{\|(n)\|} \mu^{\otimes}_{p_b}((n)) \sum_{y \in GF(2)^{|E|}} \mu^{\otimes}_p(y) \frac{I_{U_y}((n))}{Z_{n,\, p_b}(y)} \prod_{e \in \rho_{ij}} (-1)^{y_e}$$

where C_{ij} denotes the set of graphs (n) containing a cluster in which nodes $i \neq j$ are linked by a path ρ_{ij} of occupied edges. The pair correlation function is not gauge invariant. Let $\mathscr{T}$ be a linear supplement of $\mathscr{C}$. Then the inner sum becomes

$$\sum_{t \in \mathscr{T}} \frac{I_{U_t}((n))}{Z_{n,\, p_b}(t)} \sum_{c \in \mathscr{C}} \mu^{\otimes}_p(t + c)(-1)^{\langle \rho_{ij},\, t+c \rangle}$$

We assume that the path ρ_{ij} is simple. Let g_{ij} be a spanning tree of G containing ρ_{ij}. Let $\mathscr{T}_{ij}$ be the linear supplement of $\mathscr{C}$ generated by the edges of $E \backslash E(g_{ij})$ (see Section 2). By construction, $t \in \mathscr{T}_{ij} \Rightarrow t_e = 0$ $\forall e \in \rho_{ij}$. It follows that $(-1)^{\langle \rho_{ij},\, t+c \rangle} = (-1)^{\langle \rho_{ij},\, c \rangle}$, $\forall t \in \mathscr{T}_{ij}$.

Let $(\Omega, \bullet)$ be the multiplicative group with componentwise spin multiplication, and consider the group homomorphism $\Psi\colon \Omega \to \mathscr{C}$ given by $\Psi(\sigma)_{ij} = (1/2)(1 - \sigma_i \sigma_j)$. $\sum_{(ij) \in E} (-1)^{t_{ij}} \sigma_i \sigma_j = \sum_{(ij) \in E} (-1)^{(t+c)_{ij}}$, where we set $c := \Psi(\sigma)$ with $\Psi(-\sigma) = \Psi(\sigma)$. Thus $\sum_{(ij) \in E} (-1)^{t_{ij}} \sigma_i \sigma_j = |E| - 2\, |t + c|$, and it follows that $\exp(\beta_{p^*} H_t(\sigma)) = \exp(\beta_{p^*}|E|)\exp(-2\beta_{p^*}|t+c|)$. But $\mu_p^{\otimes}(t+c) = (1-p)^{|E|}(p/(1-p))^{|t+c|}$, and it follows that

$$\begin{aligned}\mu_p^{\otimes}(t+c) &= (1-p)^{|E|}\exp(-\beta_{p^*})^{|E|}\exp(\beta_{p^*}H_t(\sigma))\\ &= \sqrt{p(1-p)}^{|E|}\exp(\beta_{p^*}H_t(\sigma))\end{aligned}$$

But $(-1)^{\langle \rho_{ij}, c\rangle} = \sigma_i\sigma_j$, and we arrive at

$$\begin{aligned}\sum_{c\in\mathscr{C}}\mu_p^{\otimes}(t+c)(-1)^{\langle\rho_{ij},c\rangle} &= \tfrac{1}{2}\sqrt{p(1-p)}^{|E|}\sum_{\sigma\in\Omega}\exp(\beta_{p^*}H_t(\sigma))\,\sigma_i\sigma_j\\ &= \tfrac{1}{2}\sqrt{p(1-p)}^{|E|}Z_{p^*}(t)\langle\sigma_i\sigma_j\rangle_{\pi^t_{p^*}}\end{aligned}$$

Using (2.2), we finally obtain that

$$\sum_{c\in\mathscr{C}}\mu_p^{\otimes}(t+c)(-1)^{\langle\rho_{ij},c\rangle} = \bar{\mu}_p^{\otimes}(t)\langle\sigma_i\sigma_j\rangle_{\pi^t_{p^*}}$$

giving the required formula.

Assume that $0 \leqslant p \leqslant 1/2$. $|\langle\sigma_i\sigma_j\rangle_{\pi^t_{p^*}}| \leqslant \mu^t_{p^*}(i\rightsquigarrow j) \leqslant \langle\sigma_i\sigma_j\rangle_{\pi^0_{p^*}}$, where the last inequality holds since $\mu^t_{p^*}$ is stochastically dominated by $\mu^0_{p^*}$ and the event $\{i\rightsquigarrow j\}$ is increasing (see N. (1994)). Thus

$$|\mathrm{Ave}_{\otimes P}(\langle\sigma_i\sigma_j\rangle_{\pi^y_{\beta_b}})| \leqslant \langle\sigma_i\sigma_j\rangle_{\pi^0_{p^*}}\sum_{(n)\in C_{ij}}2^{|(n)|}\mu^{\otimes}_{p_b}((n))\sum_{t\in\mathscr{T}_{ij}}\frac{I_{U_t}((n))}{Z_{n,\,p_b}(t)}\bar{\mu}_p^{\otimes}(t)$$

and the inequality follows from the gauge invariance of $I_{U_y}((n))$ and $Z_{n,\,p_b}(y)$. When $1/2 \leqslant p < 1$, we can write $|\langle\sigma_i\sigma_j\rangle_{\pi^t_{p^*}}| = |\langle\sigma_i\sigma_j\rangle_{\pi^{t+1}_{(1-p)^*}}|$, where $1 = (1,..., 1) \in GF(2)^{|E|}$, $\beta_{(1-p)^*} = -\beta_{p^*}$, and thus $|\langle\sigma_i\sigma_j\rangle_{\pi^t_{p^*}}| \leqslant \langle\sigma_i\sigma_j\rangle_{\pi^0_{(1-p)^*}}$, giving the final result.

Remark 1. Let $G = \mathbf{Z}^2$. We get that $\mathrm{Ave}_{\otimes P}(\langle\sigma_i\sigma_j\rangle_{\pi^y_{\beta_b}}) \to 0$ when $\langle\sigma_i\sigma_j\rangle_{\pi^0_{p^*}} \to 0$, $\|i-j\| \to \infty$, that is when $\tanh(\beta_{p^*}) = 1-2p < \sqrt{2}-1$, i.e., $p_0 < p \leqslant 1/2$, where $p_0 := 1 - 1/\sqrt{2}$ is the critical parameter associated with the pair dissociation phase transition (see Schuster (1979), Kolan and Palmer (1981) or Liebmann and Schuster (1981), and the related discussion in Binder and Young (1986)).

A high temperature expansion of $\langle\sigma_i\sigma_j\rangle_{\pi^t_{p^*}}$ can be derived as in the ferromagnetic case. It is perhaps interesting to see how one can obtain such a representation by using tools of coding theory, which deal implicitely with the Fourier transform on linear subspaces of $GF(2)^{|E|}$. To this purpose, consider the sum

$$\sum_{c \in \mathscr{C}} \mu_p^{\otimes}(t+c)(-1)^{\langle \rho_{ij}, c\rangle} = (1-p)^{|E|}\left(2 \sum_{c \in \mathscr{C}_p} z_p^{|t+c|} - \sum_{c \in \mathscr{C}} z_p^{|t+c|}\right), \qquad \rho := \rho_{ij}$$

where $\mathscr{C}_p$ denotes the subgroup of $\mathscr{C}$ consisting in those cuts c which are orthogonal to ρ_{ij}, that is $\langle c, \rho_{ij}\rangle = 0$. Given any linear code $\mathscr{L}$ and $t \in GF(2)^{|E|}$, $t \notin \mathscr{L}$, consider the augmented code $\mathscr{L}_t$ given by the direct sum $\mathscr{L} \oplus \langle t \rangle$, where $\langle t \rangle$ is the linear code generated by the singleton $\{t\}$. Then $\sum_{c \in \mathscr{C}_p} z_p^{|t+c|} = W_{(\mathscr{C}_p)_t}(z_p) - W_{\mathscr{C}_p}(z_p)$, and $\sum_{c \in \mathscr{C}} z_p^{|t+c|} = W_{\mathscr{C}_t}(z_p) - W_{\mathscr{C}}(z_p)$. Using MacWilliams identity (see (2.3)) we obtain that the inner sum is given by

$$\begin{gathered} 2^{-|E|}(2(|(\mathscr{C}_p)_t|\, W_{(\mathscr{C}_p)_t^\perp}(1-2p) - |\mathscr{C}_p|\, W_{\mathscr{C}_p^\perp}(1-2p)) \\ -(|\mathscr{C}_t|\, W_{\mathscr{C}_t^\perp}(1-2p) - |\mathscr{C}|\, W_{\mathscr{C}^\perp}(1-2p))) \end{gathered}$$

By construction, $|(\mathscr{C}_p)_t| = 2\,|\mathscr{C}_p|$ and $|\mathscr{C}_t| = 2\,|\mathscr{C}|$. Concerning the size $|\mathscr{C}_p|$, we need an argument: $\mathscr{C}_p$ consists in all cuts having an even number of edges situated on ρ_{ij}. We can limit the study by considering only sums of elementary cuts $c(k)$ (see Section 2) with $k \in V(\rho_{ij})$, where $V(\rho_{ij})$ is the node set of the path ρ_{ij}. On such a path, the restricted cut code is the full binary space $GF(2)^{|E(\rho_{ij})|}$, and the number of words of even weight is equal to the number of words of odd weight, and it follows that $2\,|\mathscr{C}_p| = |\mathscr{C}|$. We get then the equivalent formula

$$\frac{|\mathscr{C}|}{2^{|E|}}\,(2(W_{(\mathscr{C}_p)_t^\perp}(1-2p) - W_{\mathscr{C}_t^\perp}(1-2p)) - (W_{\mathscr{C}_p^\perp}(1-2p) - W_{\mathscr{C}^\perp}(1-2p))) \tag{3.1}$$

By construction, $\mathscr{C}_p^\perp$ consists in all $w \in GF(2)^{|E|}$ such that $\langle w, c\rangle = 0\ \forall c \in \mathscr{C}_p$. But every elementary cut $c(k)$, with k situated on the path ρ_{ij} but different of the ends i and j is in $\mathscr{C}_p$. Every element w of the dual of $\mathscr{C}_p$ must have even degree at these nodes, since $\langle w, c(k)\rangle = \sum_{j:(jk) \in E} w_{jk} = 0 \bmod 2$. For nodes situated in $V \backslash V(\rho_{ij})$, the same result holds. It follows that $\mathscr{C}_p^\perp \backslash \mathscr{C}^\perp$ consists in all subgraphs w having even degree at nodes different from i and j, but at least one node i or j with odd degree. Any graph with these properties must have odd degree at i and j ($\sum_k \deg(k) = 2\,|E|$). Concerning the augmented codes, we have $(\mathscr{C}_p)_t^\perp = \{w \in \mathscr{C}_p^\perp;\ \langle w, t\rangle = 0\}$, and $\mathscr{C}_t^\perp = \{w \in \mathscr{C}^\perp;\ \langle w, t\rangle = 0\}$. It follows that $(\mathscr{C}_p)_t^\perp \backslash \mathscr{C}_t^\perp$ consists in all graphs of $\mathscr{C}_p^\perp \backslash \mathscr{C}^\perp$ which are orthogonal to t, and thus consists in all subgraphs with even degree at nodes different from i and j, odd degrees at i and j, and orthogonal to t, that is containing an even number of edges of t. On finally obtain the expansion

$$\langle \sigma_i \sigma_j \rangle_{\pi^t_{p*}} = \left(\sum_{w \in \mathscr{C}_p^{\perp} \setminus \mathscr{C}^{\perp}} \tanh(\beta_{p*})^{|w|} (-1)^{\langle w, t \rangle} \right) \times \left(\sum_{w \in \mathscr{C}^{\perp}} \tanh(\beta_{p*})^{|w|} (-1)^{\langle w, t \rangle} \right)^{-1}$$

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